XV.—*On the Properties of Inextensible Surfaces.* By the Rev. JOHN H. JELLETT, A. M., Fellow of Trinity College, and Professor of Natural Philosophy in the University of Dublin.

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1. ALTHOUGH the celebrated theorems of GAUSS have received from mathematicians much and deserved attention, inducing them to bestow considerable labour upon obtaining for these theorems simple and elegant demonstrations I do not find that any attempt has been made to extend his discoveries upon this subject. Yet the highly interesting character of the theorems alluded to might naturally induce the expectation of other important results connected with the theory of inextensible surfaces, sufficient to repay the labour of a more general consideration of the question than has been (so far as I am aware) as yet attempted. I propose, therefore, in the present Memoir to consider generally what are the conditions to which the displacements of a continuous inextensible membrane are subject. These conditions are expressed (as will be seen) by a system of three partial differential equations of a very simple form, which contain the solution of all questions connected with this theory. From these equations I shall deduce general expressions for the variations which the differential coefficients,

$$\frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dx^2}, \frac{d^2z}{dxdy}, \frac{d^2z}{dy^2},$$

undergo in consequence of the displacement of the membrane. These expressions give immediately the two theorems of GAUSS. I shall then proceed to consider how far the flexibility of the membrane is destroyed by rendering rigid any curve traced upon its surface. I shall in the next place investigate the laws which govern the displacement of a surface which is partially exten-

sible (as hereafter explained), and how far the preceding theorems are applicable to such surfaces; and, finally, I shall consider how far these conclusions are applicable to the laminæ which we find in nature, which are neither wholly inextensible nor wholly devoid of thickness. The results arrived at will be found, I think, sufficiently remarkable to attract the attention of mathematicians to this subject.

2. *Definition of an Inextensible Surface.*—Two definitions of inextensibility have been given by LAGRANGE and GAUSS respectively. According to the former, who defines the force which resists extension to be the force which resists the increase of superficial area, a surface is inextensible if it be impossible to change the superficial area of any portion of it. But this definition seems to be hardly consistent with the meaning ordinarily attached to the word "inextensible." For if we conceive a membrane admitting of being indefinitely extended in any direction, but of such a nature, that an extension in any one direction is always accompanied by a corresponding contraction in another, so as to preserve the area unchanged, such a membrane would be, according to LAGRANGE's definition, *inextensible.* But it appears more consistent with ordinary ideas to consider an inextensible surface to be one which does not admit of *any* extension, rather than one whose capacities of extension and contraction counterbalance one another in the manner above described. I shall, therefore, in the present Memoir adopt the definition of GAUSS, as more exactly embodying the ordinary ideas on the subject, adding to it the definition of partially extensible surfaces, a class not noticed by GAUSS, but presenting some remarkable properties. These definitions are as follows:

I. *A surface is said to be inextensible, when the length of a curve traced arbitrarily upon it is unchangeable by any force which can be applied to it.*

II. *A surface is partially extensible, if there be at each of its points one or more inextensible directions; in other words, if it be possible to trace at each point one or more inextensible curves.*

We shall now proceed to consider how these definitions may be mathematically expressed, commencing with the case of inextensible surfaces.

3. *Deduction of the Equations which connect the Displacements of an Inextensible Surface.* Let ds be the element of a curve traced in any direction upon the surface, and let δ be the symbol of displacement, i. e. a symbol denoting the

passage of a molecule, or physical point, from one geometrical point of space to another. Then, since the curve of which ds is an element, is by the assumed definition inextensible, we must have

$$\delta ds = 0;$$

or, putting for ds its value,

$$\sqrt{(dx^2 + dy^2 + dz^2)};$$

and performing the operations indicated by δ,

$$dx d\delta x + dy d\delta y + dz d\delta z = 0; \tag{A}$$

recollecting that δ is a commutative symbol. But since the displacements δx, δy, δz, refer to a point on the surface, we must have

$$d\delta x = \frac{d\delta x}{dx}dx + \frac{d\delta x}{dy}dy,$$

$$d\delta y = \frac{d\delta y}{dx}dx + \frac{d\delta y}{dy}dy,$$

$$d\delta z = \frac{d\delta z}{dx}dx + \frac{d\delta z}{dy}dy;$$

x, y, being the independent variables.

Let

$$dz = pdx + qdy,$$

be the equation of the surface.

Substituting for $d\delta x$, $d\delta y$, $d\delta z$, dz, in equation (A), we have

$$\left(\frac{d\delta x}{dx} + p\frac{d\delta z}{dx}\right)dx^2 + \left(\frac{d\delta y}{dx} + \frac{d\delta x}{dy} + q\frac{d\delta z}{dx} + p\frac{d\delta z}{dy}\right)dxdy + \left(\frac{d\delta y}{dy} + q\frac{d\delta z}{dy}\right)dy^2 = 0.$$

But since the condition expressed in this equation is supposed to hold for *every* curve traced upon the surface, it must be true for all values of

$$\frac{dy}{dx}.$$

We have, therefore,

$$\begin{aligned} \frac{d\delta x}{dx} + p\frac{d\delta z}{dx} &= 0, \\ \frac{d\delta y}{dx} + \frac{d\delta x}{dy} + q\frac{d\delta z}{dx} + p\frac{d\delta z}{dy} &= 0, \\ \frac{d\delta y}{dy} + q\frac{d\delta z}{dy} &= 0. \end{aligned} \tag{B}$$

These equations may be put under a somewhat simpler form, by assuming

$$u = \delta x + p\delta z,$$
$$v = \delta y + q\delta z,$$
$$w = \delta z.$$

Making these substitutions, we find

$$\left.\begin{aligned} \frac{du}{dx} - wr &= 0, \\ \frac{du}{dy} + \frac{dv}{dx} - 2ws &= 0, \\ \frac{dv}{dy} - wt &= 0; \end{aligned}\right\} \quad \text{(C)}$$

where r, s, t, are used in their ordinary sense to denote the differential coefficients

$$\frac{d^2z}{dx^2}, \quad \frac{d^2z}{dxdy}, \quad \frac{d^2z}{dy^2};$$

derived from the equation of the surface. Any one of the quantities u, v, w, may be determined by means of a differential equation of the second order. Thus, for example, eliminating w between the equations (C), we find,

$$\frac{1}{r}\frac{du}{dx} = \frac{1}{2s}\left(\frac{du}{dy} + \frac{dv}{dx}\right) = \frac{1}{t}\frac{dv}{dy}.$$

Hence,

$$\frac{dv}{dx} = \frac{2s}{r}\frac{du}{dx} - \frac{du}{dy},$$
$$\frac{dv}{dy} = \frac{t}{r}\frac{du}{dx}.$$

Differentiating the first of these equations with respect to y, and the second with respect to x, and subtracting, we find easily,

$$r\frac{d^2u}{dy^2} - 2s\frac{d^2u}{dxdy} + t\frac{d^2u}{dx^2} = \frac{1}{r}\frac{d(rt - s^2)}{dx}\frac{du}{dx}; \quad \text{(D)}$$

and similarly for v,

$$r\frac{d^2v}{dy^2} - 2s\frac{d^2v}{dxdy} + t\frac{d^2v}{dx^2} = \frac{1}{t}\frac{d(rt - s^2)}{dy}\frac{dv}{dy}. \quad \text{(E)}$$

The equation for w may readily be deduced from (C). Differentiating the first of these equations with respect to y, and the second with respect to x, and subtracting, we have

$$\frac{d^2v}{dx^2} = 2s\frac{dw}{dx} - r\frac{dw}{dy} + w\frac{ds}{dx}.$$

Differentiating this equation with respect to y,

$$\frac{d^3v}{dx^2dy} = 2s\frac{d^2w}{dxdy} - r\frac{d^2w}{dy^2} + 2\frac{ds}{dy}\frac{dw}{dx} + w\frac{d^2s}{dxdy}.$$

Again, differentiating the third of equations (C) twice with regard to x,

$$\frac{d^3v}{dx^2dy} = t\frac{d^2w}{dx^2} + 2\frac{dt}{dx}\frac{dw}{dx} + w\frac{d^2t}{dx^2}.$$

Subtracting these equations one from the other, we find,

$$r\frac{d^2w}{dy^2} - 2s\frac{d^2w}{dxdy} + t\frac{d^2w}{dx^2} = 0. \qquad \text{(F)}$$

Some interesting results followed at once from the fundamental equations. Thus, for example, if the displacements of the surface be all parallel to the same plane, we shall have, taking this plane for the plane of xy,

$$w = 0.$$

The equations (C) are thus reduced to

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} + \frac{dv}{dx} = 0, \quad \frac{dv}{dy} = 0.$$

Integrating this system of equations, we find, without difficulty,

$$u = A + By, \qquad v = C - Bx;$$

or since $\delta z = 0$

$$\delta x = A + By, \quad \delta y = C - Bx;$$

A, B, C, being constants. These equations express the following theorem:

If the displacements of an inextensible surface be all parallel to the same plane, the surface moves as a rigid body.

More generally, if we make

$$w = \delta z = ax - by + e,$$

in the equations (C) we shall find without difficulty the solution:

$$\delta x = cy - az + e',$$
$$\delta y = bz - cx + e''.$$

Hence we infer that—

If the movement of an inextensible surface, parallel to any one line, be that of a rigid body, the entire movement is that of a rigid body.

4. *Variations of the Differential Coefficients.*—If we denote by δ' the variation, properly so called, i. e., the change which the function receives in consequence of a change of form, it is evident that

$$\delta z = p\delta x + q\delta y + \delta z',$$

$$\left.\begin{aligned}\delta p &= \frac{dp}{dx}\,\delta x + \frac{dp}{dy}\,\delta y + \frac{d\delta z'}{dx} = \frac{dp}{dx}\,\delta x + \frac{dp}{dy}\,\delta y + \frac{d\,(\delta z - p\delta x - q\delta y)}{dx}\\ &= \frac{d\delta z}{dx} - p\,\frac{d\delta x}{dx} - q\,\frac{d\delta y}{dx},\\ \delta q &= \frac{d\delta z}{dy} - p\,\frac{d\delta x}{dy} - q\,\frac{d\delta y}{dy}.\end{aligned}\right\}\quad (G)$$

Eliminating

$$\frac{d\delta x}{dx}, \quad \frac{d\delta y}{dy},$$

from these equations, by means of the first and third of equations (B), we have

$$\delta p = (1 + p^2 + q^2)\,\frac{d\delta z}{dx} - q\left(\frac{d\delta y}{dx} + q\,\frac{d\delta z}{dx}\right),$$

$$\delta q = (1 + p^2 + q^2)\,\frac{d\delta z}{dy} - p\left(\frac{d\delta x}{dy} + p\,\frac{d\delta z}{dy}\right).$$

In the same way we find,

$$\delta r = \frac{d\delta p}{dx} - r\,\frac{d\delta x}{dx} - s\,\frac{d\delta y}{dx},$$

$$\delta s = \frac{d\delta p}{dy} - r\,\frac{d\delta x}{dy} - s\,\frac{d\delta y}{dy} = \frac{d\delta q}{dx} - s\,\frac{d\delta x}{dx} - t\,\frac{d\delta y}{dx},$$

$$\delta t = \frac{d\delta q}{dy} - s\,\frac{d\delta x}{dy} - t\,\frac{d\delta y}{dy}.$$

Substituting for δp, δq, and reducing the resulting expressions by means of equations (D), (E), (F), we find ultimately,

$$\delta r = (1+p^2+q^2)\frac{d^2\delta z}{dx^2} - r\left(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\right) - 2\left(r\frac{d\delta x}{dx}+s\frac{d\delta y}{dx}\right),$$

$$\delta s = (1+p^2+q^2)\frac{d^2\delta z}{dxdy} - 2s\left(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\right) - \left(r\frac{d\delta x}{dy}+t\frac{d\delta y}{dx}\right), \qquad \text{(H)}$$

$$\delta t = (1+p^2+q^2)\frac{d^2\delta z}{dy^2} - t\left(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\right) - 2\left(s\frac{d\delta x}{dy}+t\frac{d\delta y}{dy}\right).$$

From the two equations (G) it is easy to verify that the element of the superficial area remains constant; for if we multiply the first of these equations by p, and the second by q, and add them, we find, recollecting the second of equations (B),

$$p\delta p + q\delta q = (1+p^2+q^2)\left(p\frac{d\delta z}{dx}+q\frac{d\delta z}{dy}\right);$$

or from the first and third of equations (B),

$$p\delta p + q\delta q + (1+p^2+q^2)\left(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\right) = 0, \qquad \text{(I)}$$

which is obviously equivalent to

$$\delta\sqrt{(1+p^2+q^2)}\,dxdy = 0.$$

Again, multiplying the first of equations (H) by t, the second by $2s$, and the third by r, and subtracting the second product from the sum of the other two, we have

$$t\delta r + r\delta t - 2s\delta s = \delta(rt-s^2)$$

$$= (1+p^2+q^2)\left(t\frac{d^2\delta z}{dx^2} - 2s\frac{d^2\delta z}{dxdy} + r\frac{d^2\delta z}{dy^2}\right) - 4(rt-s^2)\left(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\right)$$

$$= -4(rt-s^2)\left(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\right).$$

Hence, and from equation (I), it is easy to see that

$$\frac{\delta(rt-s^2)}{rt-s^2} - 4\frac{(p\delta p+q\delta q)}{1+p^2+q^2} = 0,$$

which is plainly equivalent to

$$\delta \frac{rt - s^2}{(1 + p^2 + q^2)^2} = \delta \frac{1}{RR'} = 0. \qquad \text{(K)}$$

This equation is the analytical statement of GAUSS's celebrated theorem, namely, that

In all the possible movements of an inextensible surface, the product of the principal radii of curvature at every point of the surface is constant.

Let S be a portion of the surface bounded by any closed curve. Conceive this curve to be referred to the surface of a sphere, by radii drawn parallel to the normals, and let S' be the included portion of the spherical surface. Then, if the radius of the sphere be supposed to be unity,

$$S' = \iint \frac{dS}{RR'},$$

and therefore,

$$\delta S' = \iint \delta \frac{dS}{RR'} = 0.$$

Hence, *In all possible motions of an inextensible surface, the area of the spherical curve corresponding to any closed curve described upon the surface* (denominated by GAUSS the "curvatura integra") *remains constant.*

This is the second theorem of GAUSS.

5. We shall next proceed to consider the effect of *fixing* any curve upon the surface. The determination of the displacement of the surface in this case will obviously depend upon the following analytical problem :—" To find three functions u, v, w, which shall satisfy the partial differential equations,

$$\frac{du}{dx} - wr = 0,$$

$$\frac{du}{dy} + \frac{dv}{dx} - 2ws = 0,$$

$$\frac{dv}{dy} - wt = 0,$$

and shall, moreover, have the values

$$u = 0, \quad v = 0, \quad w = 0,$$

for all points of a given curve or portion of a curve."

Let

$$dy = mdx,$$

be the equation of the projection of the given curve upon the plane of xy. Then since u, v, w, vanish for a continuous portion of this curve, we must have

$$\begin{aligned} \frac{du}{dx} + m\frac{du}{dy} &= 0, \\ \frac{dv}{dx} + m\frac{dv}{dy} &= 0, \\ \frac{dw}{dx} + m\frac{dw}{dy} &= 0. \end{aligned} \tag{L}$$

But if we make

$$w = 0$$

in the first and third of equations (C), we shall have

$$\frac{du}{dx} = 0, \quad \frac{dv}{dy} = 0.$$

Hence and from equations (L) we have

$$\frac{du}{dy} = 0, \quad \frac{dv}{dx} = 0.$$

Differentiating these equations upon the same principle, we have

$$\begin{aligned} \frac{d^2u}{dx^2} + m\frac{d^2u}{dxdy} &= 0, & \frac{d^2u}{dxdy} + m\frac{d^2u}{dy^2} &= 0, \\ \frac{d^2v}{dx^2} + m\frac{d^2v}{dxdy} &= 0, & \frac{d^2v}{dxdy} + m\frac{d^2v}{dy^2} &= 0. \end{aligned} \tag{M}$$

Hence it is easily seen that the equations (D) and (E), p. 346, become for this curve

$$\begin{aligned} (r + 2sm + tm^2)\frac{d^2u}{dxdy} &= 0, \\ (r + 2sm + tm^2)\frac{d^2v}{dxdy} &= 0. \end{aligned} \tag{N}$$

Hitherto the reasoning employed has been perfectly general, embracing surfaces of every class. But in our subsequent investigations we must discuss

severally the three great classes into which surfaces are divided with respect to their curvature, namely :

1. Surfaces whose principal curvatures are *similar*, or those in which

$$rt - s^2 > 0.$$

2. Developable surfaces, in which

$$rt - s^2 = 0.$$

3. Surfaces whose principal curvatures are *dissimilar*, or those in which

$$rt - s^2 < 0.$$

I. *Surfaces whose principal curvatures are similar.*—In this case it is plain that the equation

$$r + 2sm + tm^2 = 0$$

is impossible, whatever be the value of m. Therefore the equations (N) can only be satisfied by making

$$\frac{d^2u}{dxdy} = 0, \qquad \frac{d^2v}{dxdy} = 0.$$

Hence, and from equations (M), we find

$$\left.\begin{aligned} &\frac{d^2u}{dx^2} = 0, \qquad \frac{d^2u}{dxdy} = 0, \qquad \frac{d^2u}{dy^2} = 0, \\ &\frac{d^2v}{dx^2} = 0, \qquad \frac{d^2v}{dxdy} = 0, \qquad \frac{d^2v}{dy^2} = 0, \end{aligned}\right\} \qquad (\mathrm{O})$$

which must hold throughout the fixed curve. Again, differentiating equations (D) and (E) (which are true generally) with regard to x, and rejecting differential coefficients of the first and second order, which vanish for the fixed curve, we have

$$\left.\begin{aligned} &r\frac{d^3u}{dxdy^2} - 2s\frac{d^3u}{dx^2dy} + t\frac{d^3u}{dx^3} = 0, \\ &r\frac{d^3v}{dxdy^2} - 2s\frac{d^3v}{dx^2dy} + t\frac{d^3u}{dx^3} = 0, \end{aligned}\right\} \qquad (\mathrm{P})$$

which must hold throughout the fixed curve.

Differentiating equations (M) as before, and eliminating

$$\frac{d^3u}{dx^2dy}, \qquad \frac{d^3u}{dx^3},$$

from equations (P), we have

$$\left.\begin{aligned}(r + 2sm + tm^2)\,\frac{d^3u}{dxdy^2} &= 0,\\(r + 2sm + tm^2)\,\frac{d^3v}{dxdy^2} &= 0.\end{aligned}\right\} \qquad \text{(Q)}$$

Hence it is easy to infer, as before, that all the differential coefficients of the third order vanish for points of the surface situated on the fixed curve; and a very slight examination will show that by proceeding in the same manner we shall find that *all* the differential coefficients of u, of all orders, vanish for the limiting curve. Now if u be a function of the same form throughout the surface, it is plain that these conditions can only be satisfied by the supposition that u vanishes at every point. The same conclusion will hold if u change its form. For if u be supposed to have the same form for all points between the limiting curve and any other curve drawn arbitrarily, it is plain, from what has been said, that its value can be no other than zero. Hence for all points of the second curve

$$u = 0.$$

Now it is evident that the same reasoning which was before applied to the limiting curve is equally applicable to this second curve, and so on for any number of curves bounding those parts of the surface for which the form of u is the same. It appears, therefore, from the foregoing reasoning, that we must have throughout the entire surface

$$u = 0.$$

By precisely similar reasoning it may be shown that we have throughout the entire surface

$$v = 0;$$

and on referring to equations (C), it will be seen that it follows at once from these equations that

$$w = 0.$$

Replacing u, v, w, by their values in terms of δx, δy, δz, we have for every point of the surface

$$\delta x = 0, \quad \delta y = 0, \quad \delta z = 0.$$

Hence we infer the following theorem :

If any curve be traced upon an inextensible surface, whose principal curvatures are finite and of the same sign, and if this curve be rendered immovable, the entire surface will become immovable also.

More generally, let it be required to determine a system of values of u, v, w, which shall satisfy the equations (C), and which shall have at all points of a given curve, or part of a curve, the given values

$$u = u_1, \quad v = v_1, \quad w = w_1.$$

Then it is easy to show from the foregoing discussion, that there is but one such system.

For, if possible, let there be two systems of values,

$$u = U, \quad v = V, \quad w = W,$$
$$u = U', \quad v = V', \quad w = W',$$

which satisfy the given conditions. Then since the equations (C), which these two systems of values are supposed to satisfy, are linear, it is plain that if we form a third system,

$$u = U - U', \quad v = V - V', \quad w = W - W',$$

this system will also satisfy equations (C). But as the values of u, v, w are given for the limiting curve, the two assumed systems must be coincident throughout this curve, and therefore we must have for all its points,

$$U - U' = 0, \quad V - V' = 0, \quad W - W' = 0.$$

Now we have seen in the foregoing discussion that if u, v, w be a system of values satisfying these two conditions, we must have generally

$$u = 0, \quad v = 0, \quad w = 0.$$

Hence it is plain that at every point of the surface

$$U - U' = 0, \quad V - V' = 0, \quad W - W' = 0.$$

The two systems of values are therefore identical. Hence we infer the theorem—

If a curve be traced upon an inextensible surface, whose principal curvatures are finite, and of the same sign, and if any given determinate motion be assigned to this curve, the motion of the entire surface is determinate and unique.

Thus, for example, it is easily shown that if the limiting curve be made *rigid*, the entire surface will become rigid also.

II. We shall next consider the case of *developable* surfaces, or those in which

$$rt - s^2 = 0.$$

This case may be subdivided into two, which require to be considered separately. These cases are—

1. When the fixed curve is either a rectilinear section of the surface or the *arête de rebroussement.*

2. When the fixed curve is not either of these.

1. Let the fixed curve be a rectilinear section. Then it is plain that this curve must satisfy the equation

$$r + 2sm + tm^2 = 0,$$

which expresses the fact that the radius of curvature of the normal section passing through this line, i. e. in the case of a developable surface, of the line itself, is infinite.

Hence the equations (N) become identically true, without supposing that

$$\frac{d^2u}{dxdy} = 0, \qquad \frac{d^2v}{dxdy} = 0.$$

It is plain, therefore, that the reasoning by which it was shown that the several differential coefficients of u, v vanish for the fixed curve, is no longer applicable, and that the several conditions of the problem may be satisfied *without* supposing u, v to vanish at every point of the surface. We infer, therefore, that,

In a developable surface composed of an inextensible membrane, any one of its rectilinear sections may be fixed without destroying the flexibility of the membrane.

And it is easily seen that the same conclusion will hold if the fixed curve be the *arête de rebroussement* of the developable surface.

2. Let the fixed curve be neither a right line nor the *arête de rebroussement.* Then since this curve does not satisfy the equation

$$r + 2sm + tm^2 = 0,$$

we must have, as in the first case,

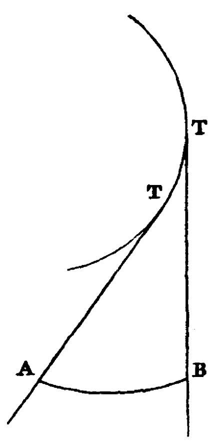
$$\frac{d^2u}{dxdy} = 0, \qquad \frac{d^2v}{dxdy} = 0.$$

All the reasoning of that case is therefore strictly applicable, and it will appear, as before, that *all* the differential coefficients of u must vanish for the limiting curve. Hence, if u preserve the same form, it can have no value but zero. Now let it be supposed that u may change its form ; then it is easily seen that the zero value of u can only change in passing across a curve whose equation is

$$r + 2sm + tm^2 = 0.$$

Every part of the surface, therefore, which can be reached from the fixed curve *without* crossing either the *arête de rebroussement* or a rectilinear section, is necessarily fixed. The remainder of the surface is capable of motion. Hence we have the following construction :

Let AB be a fixed curve drawn on the given membrane. Draw through the extreme points A, B, the rectilinear sections of the developable surface, and produce them to touch the *arête de rebroussement.* Then it is evident, from the foregoing analysis, that all that part of the surface which lies between the two lines, and on the same side of the *arête de rebroussement* with the fixed curve, will itself be fixed. Beyond these lines the surface is flexible.



To determine more accurately the nature of the motion of which the surface is capable, we shall now proceed to integrate the equation (D), which, for a developable surface, is in general possible.

Since $rt - s^2 = 0$, equation (D) becomes in the present case

$$r\frac{d^2u}{dy^2} - 2s\frac{d^2u}{dxdy} + t\frac{d^2u}{dx^2} = 0. \tag{R}$$

The equations of the characteristic are therefore

$$r + 2s\frac{dy}{dx} + t\frac{dy^2}{dx^2} = 0, \tag{S}$$

$$rd\,.\frac{du}{dy} + t\frac{dy}{dx}.\,d\,.\frac{du}{dx} = 0. \tag{T}$$

Let

$$p = Q$$

be the equation of the given surface; then

$$r = Q's, \qquad s = Q't,$$

where

$$Q' = \frac{dQ}{dq}.$$

Substituting these values in equation (S), we find easily

$$\frac{dy}{dx} = -\,Q'.$$

Now since equation (S) represents a rectilinear section of the surface, it is evident that in this equation Q' must be constant. Hence it becomes

$$y + Q'x = \text{const.}$$

Again, substituting for r, t, and $\frac{dy}{dx}$ in equation (T), and integrating, we find

$$Q'\frac{du}{dy} - \frac{du}{dx} = \text{const.}$$

Then the integral of equation (R), which may readily be obtained in the ordinary way, will be

$$u = xf(q) + F(q). \tag{V}$$

The following general property of the motion may be deduced from this equation:

*The rectilinear sections of the surface are rigid.**

For, since in a developable surface q is constant for the same rectilinear section, the value of u for such a section will be

$$u = Ax + B,$$

A, B being constants.

* It is easily seen, however, that these sections may all bend at the *arête de rebroussement.*

Similarly we shall have

$$v = A'x + B'.$$
$$w = A''x + B''.$$

Hence it is easy to see that if x', y', z' be the co-ordinates of the new position of the point x, y, z, we shall have

$$x' = ax + b,$$
$$y' = a'x + b',$$
$$z' = a''x + b'',$$

where a, a', a'', b, b', b'', are constant for the same rectilinear section. From these equations it is plain that the locus of the points x', y', z' is still a right line.

III. *Concavo-convex surfaces, or those in which*

$$rt - s^2 < 0.$$

It is a well-known property of surfaces of this class that at each point of the surface there are two real directions satisfying the condition

$$r \cos^2 \alpha + 2s \cos \alpha \cos \beta + t \cos^2 \beta = 0 ; \qquad \text{(W)}$$

an equation which expresses the geometrical fact, that the normal section which passes through either of these directions will have at that point an infinite radius of curvature. We may therefore conceive the entire surface to be crossed by two series of curves, such that a tangent drawn to either of them at any point shall possess this geometrical property. These curves we shall denominate (for a reason which will appear subsequently) *curves of flexure.* We shall consider separately (as before for developable surfaces) the two different cases which arise, according as the fixed curve is or is not a curve of flexure.

1. When the fixed curve is a curve of flexure it is evident, as in the case of developable surfaces, that the equation

$$(r + 2sm + tm^2) \frac{d^2u}{dxdy} = 0$$

becomes identically true *without* supposing

$$\frac{d^2u}{dxdy} = 0.$$

We conclude, therefore, as before, that any one of these curves may be fixed without destroying the flexibility of the surface. The reason for the name "curve of flexure" is thus explained. In fact we see that these curves, when fixed, allow the surface to *bend* round them, the flexure commencing at the curve itself. We shall presently show that this property is peculiar to the curves of flexure as above defined.

2. When the fixed curve is not a curve of flexure, the reasoning before given in the case of developable surfaces will show that δx, δy, δz, and all their differential coefficients, vanish for the fixed curve. If, therefore, these functions retained throughout the same form it is plain that the value of each could be no other than zero. Before proceeding to consider how far this conclusion is modified by a change in the forms of the functions, we shall prove the following theorems, which are essential to our purpose.

(I.) If the functions which represent the displacements of an inextensible surface have different forms at different points of the surface, the parts of the surface for which these functions retain the same forms are bounded by curves of flexure.

This theorem is proved by reasoning nearly identical with that of p. 354. For, if possible, let the forms of these functions change in passing across a curve which is not a curve of flexure. Let

$$u = U, \qquad v = V, \qquad w = W,$$

be the values which hold at one side of the curve, and

$$u = U', \qquad v = V', \qquad w = W',$$

those which hold at the other. Then it will appear precisely as in p. 354 that if we form a third system of values,

$$u = U - U', \qquad v = V - V', \qquad w = W - W',$$

this system will satisfy the equations (C), and will, moreover, be such that for every point of the curve in question we shall have

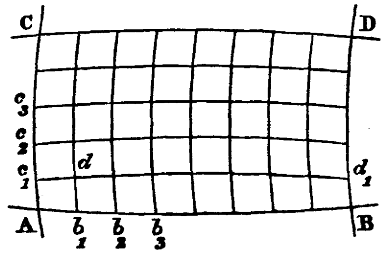
$$u = 0, \qquad v = 0, \qquad w = 0.$$

Since, then, the bounding curve is not a curve of flexure, and since U, V, W, U', V', W', are functions of determinate form, it is plain that we must have generally

$$U - U' = 0, \qquad V - V' = 0, \qquad W - W' = 0.$$

No one, therefore, of the functions u, v, w, can change its form, except in passing across a curve of flexure. Hence the proposition is evident.

(II.) Let AB, AC be two arcs of curves of flexure commencing at the same point A. Through B, C draw the curves of flexure BD, CD, meeting in D. Then if AB, AC be fixed, the entire quadrilateral ABDC is fixed also.



The truth of this theorem is nearly evident from the theory of partial differential equations, combined with the principle laid down in (I.), but it may be strictly proved as follows :

Since u, v, w, can only change their forms in passing a curve of flexure, we may suppose them to retain the same form throughout the entire of the quadrilateral $Ab_1d_1c_1$, formed by drawing the curves of flexure b_1d, c_1d.

Let $$\theta = c, \qquad \theta' = c',$$

be the equations of the two series of curves of flexure. Then, since the functions θ, θ' satisfy the differential equations

$$r\frac{d\theta^2}{dy^2} - 2s\frac{d\theta}{dx}\frac{d\theta}{dy} + t\frac{d\theta^2}{dx^2} = 0,$$

$$r\frac{d\theta'^2}{dy^2} - 2s\frac{d\theta'}{dx}\frac{d\theta'}{dy} + t\frac{d\theta'^2}{dx^2} = 0,$$

if the independent variables x, y, be changed into θ, θ', the equation

$$r\frac{d^2w}{dy^2} - 2s\frac{d^2w}{dxdy} + t\frac{d^2w}{dx^2} = 0$$

will (as is well known) assume the form

$$\frac{d^2w}{d\theta d\theta'} + P\frac{dw}{d\theta} + Q\frac{dw}{d\theta'} = 0, \qquad \text{(X)}$$

P, Q, being functions of θ, θ'. Now since w vanishes for the curve

$$\theta = c,$$

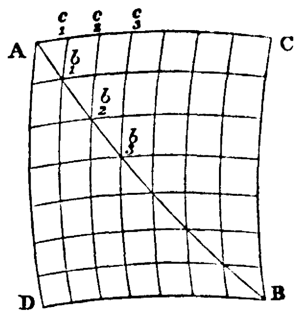
it is plain that we must have throughout this curve,

$$\frac{dw}{d\theta'} = 0, \quad \frac{d^2w}{d\theta'^2} = 0, \quad \frac{d^3w}{d\theta'^3} = 0, \ \&c.,$$

and for the curve
$$\theta' = 0,$$
$$\frac{dw}{d\theta} = 0, \quad \frac{d^2w}{d\theta^2} = 0, \quad \frac{d^3w}{d\theta^3} = 0, \text{ \&c.}$$

For the point A, therefore, which is the intersection of these curves, both these systems of equations must be satisfied. Putting, then, in equation (X),
$$\frac{dw}{d\theta} = 0, \quad \frac{dw}{d\theta'} = 0,$$
we have
$$\frac{d^2w}{d\theta d\theta'} = 0,$$
and it is easily seen that neither P nor Q will become infinite ; and by following the same reasoning with that of p. 352, we shall find that for the point A all the differential coefficients of w must vanish. Hence as the form of w remains the same throughout the quadrilateral Ab_1dc_1, we must have for the whole of that quadrilateral
$$w = 0.$$

Now it is evident that the reasoning which we have applied to the point A is in every respect applicable to b_1, and thus in succession to b_2, b_3, &c. The value of w will therefore be zero for all points of a second curve of flexure c_1d_1. And by pursuing the same method we see evidently that w must vanish throughout the whole of the quadrilateral ABDC. Hence, the direction of the axis of z being indeterminate, we shall have in general,
$$\delta z = 0, \quad \delta y = 0, \quad \delta x = 0,$$
throughout ABDC. The whole of this quadrilateral is therefore fixed. We shall now proceed to consider the general case.



Let AB be any arc of a curve (not a curve of flexure) traced upon the surface. Through A, B, draw the curves of flexure, AC, AD, BC, BD. Then if AB be fixed, the quadrilateral ACBD is fixed also.

For whatever law or laws we suppose the displacements to follow, it is plain that we may assume a number of points, b_1, b_2, b_3, &c. so close that one of these displacements, w, for example, shall retain the same form throughout each one of the quadrilaterals

Ab_1, b_1b_2, b_2b_3, &c., formed by drawing curves of flexure through b_1, b_2, &c. Hence, and from p. 359, it is evident that w must vanish throughout the entire of each of these quadrilaterals. But if

$$w = 0$$

for the quadrilaterals Ab_1, b_1b_2, it follows from Theorem II. p. 360, that w must vanish for the quadrilateral b_1c_2; and by pursuing the same method we shall easily see that we must have

$$w = 0$$

for each of the quadrilaterals into which ACBD is divided. Hence the truth of the proposition is evident. This proposition may be expressed by saying that

If an arc of a curve traced upon an inextensible surface be rendered fixed or rigid, the entire of the quadrilateral, formed by drawing the two curves of flexure through each extremity of the curve, becomes fixed or rigid also.

6. We shall now proceed to consider the case of surfaces which, without being wholly inextensible, have at each point one or more inextensible directions.

Reverting to the discussion of p. 345, and making

$$\frac{dx}{ds} = \cos\alpha, \qquad \frac{dy}{ds} = \cos\beta,$$

we find easily

$$\frac{\delta ds}{ds} = \left(\frac{du}{dx} - wr\right)\cos^2\alpha + \left(\frac{du}{dy} + \frac{dv}{dx} - 2ws\right)\cos\alpha\cos\beta + \left(\frac{dv}{dy} - wt\right)\cos^2\beta. \quad (\mathrm{Y})$$

From this equation it is plain that, unless the coefficients of

$$\cos^2\alpha, \qquad \cos\alpha\cos\beta, \qquad \cos^2\beta,$$

vanish separately, there can be, for each law of displacement, but two values of

$$\frac{\cos\alpha}{\cos\beta},$$

which will satisfy the equation

$$\delta ds = 0.$$

If these coefficients vanish separately, δds will vanish for every direction round the point. Hence it is easy to infer the following theorems :—

If a surface have at each point three or more inextensible directions, it is wholly inextensible.

A surface may have at each point one or two inextensible directions, without being wholly inextensible.

Suppose that the given surface has at each point two inextensible curves included in the equation

$$Rdx^2 + 2\,Sdxdy + Tdy^2 = 0,$$

or

$$R\cos^2 a + 2\,S\cos a\,\cos\beta + T\cos^2\beta = 0.$$

Then, as this equation must be identical with

$$\left(\frac{du}{dx} - wr\right)\cos^2 a + \left(\frac{du}{dy} + \frac{dv}{dx} - 2ws\right)\cos a\,\cos\beta + \left(\frac{dv}{dy} - wt\right)\cos^2\beta = 0,$$

we shall have

$$\frac{1}{R}\left(\frac{du}{dx} - wr\right) = \frac{1}{2S}\left(\frac{du}{dy} + \frac{dv}{dx} - 2ws\right) = \frac{1}{T}\left(\frac{dv}{dy} - wt\right). \qquad \text{(Z)}$$

These two equations contain the entire theory of the surfaces under consideration.

Suppose, for example, that the surface is one of dissimilar curvatures, and that its curves of flexure are inextensible. We have then

$$R = r, \qquad S = s, \qquad T = t,$$

and the equations (Z) become

$$\frac{1}{r}\frac{du}{dx} = \frac{1}{2s}\left(\frac{du}{dy} + \frac{dv}{dx}\right) = \frac{1}{t}\frac{dv}{dy}, \qquad \text{(A')}$$

being identical with the equations which are found by eliminating w between the general equations (C), p. 346. The displacement w remains indeterminate. From these considerations it is easy to deduce the following theorem :—

If the curves of flexure traced upon a surface with dissimilar curvatures be inextensible, the most general displacement of which the surface is capable may be found by supposing it first to move as an inextensible surface, and then to receive at each point a normal displacement of arbitrary magnitude.

Let W be an arbitrary function of x and y. Then the equations (A′) being put under the form

$$\frac{du}{dx} = Wr, \qquad \frac{du}{dy} + \frac{dv}{dx} = 2Ws, \qquad \frac{dv}{dy} = Wt,$$

the expression for the extension of any small arc ds (Y) will become

$$\delta ds = (W - w)\, ds\, (r \cos^2 \alpha + 2s \cos \alpha \cos \beta + t \cos^2 \beta).$$

Hence for the class of surfaces under consideration we infer that—

The extension of any small arc of a curve commencing at a given point, divided by the arc itself, varies inversely as the radius of curvature of the normal section which passes through it.

7. Having thus investigated the case of inextensible and partially inextensible surfaces, we should, in the next place, proceed to consider how far the results arrived at are applicable to the various membranes which we find in nature, and which are neither perfectly inextensible nor altogether devoid of thickness. But before entering upon this question we shall briefly examine the case of inextensible *bodies.*

Conceive a curve to be traced in the interior of a body, passing through the *successive* physical points or molecules a, b, c, d, &c. Suppose now that the several points of the body receive small displacements, and take the curve which is the locus of the points a, b, c, d, &c. in their new position. If the length of the second curve be equal to that of the first, and if this be true of all curves which can be so drawn, the body may be said to be inextensible. Adopting this definition, we shall have the following theorems :—

I. *Every body which is perfectly inextensible is also perfectly rigid.*

II. *Any body may, without being wholly inextensible, have at each of its points an infinite number of inextensible directions, and these directions will be situated upon a cone of the second order.*

Let δx, δy, δz, be the displacements of any point in the body, and let ds be an element of a curve, making with the axes of co-ordinates the angles α, β, γ. Then it is easily seen, that the variation of this element will be given by the equation

$$\frac{\delta ds}{ds} = \frac{d\delta x}{dx}\cos^2\alpha + \frac{d\delta y}{dy}\cos^2\beta + \frac{d\delta z}{dz}\cos^2\gamma \tag{B'}$$

$$+\left(\frac{d\delta z}{dy} + \frac{d\delta y}{dz}\right)\cos\beta\cos\gamma + \left(\frac{d\delta x}{dz} + \frac{d\delta z}{dx}\right)\cos\gamma\cos\alpha + \left(\frac{d\delta y}{dx} + \frac{d\delta x}{dy}\right)\cos\alpha\cos\beta.$$

Now if the body be inextensible, we must have for all values of α, β, γ,

$$\delta ds = 0.$$

Hence we have the six equations,

$$\frac{d\delta x}{dx} = 0, \quad \frac{d\delta y}{dy} = 0, \quad \frac{d\delta z}{dz} = 0; \tag{C'}$$

$$\frac{d\delta z}{dy} + \frac{d\delta y}{dz} = 0, \quad \frac{d\delta x}{dz} + \frac{d\delta z}{dx} = 0, \quad \frac{d\delta y}{dx} + \frac{d\delta x}{dy} = 0.$$

Integrating this system of equations, which may be effected without difficulty, we find,

$$\delta x = a + Bz - Cy,$$
$$\delta y = b + Cx - Az,$$
$$\delta z = c + Ay - Bx;$$

the well-known expression for the displacements of a rigid body. These being the most general values which δx, δy, δz admit of, the truth of the first theorem is evident.

With regard to the second theorem, if the body is so constituted that the displacements δx, δy, δz must satisfy the equations

$$\frac{d\delta x}{dx} = A, \quad \frac{d\delta y}{dy} = B, \quad \frac{d\delta z}{dz} = C, \tag{D'}$$

$$\frac{d\delta z}{dy} + \frac{d\delta y}{dz} = 2A', \quad \frac{d\delta x}{dz} + \frac{d\delta z}{dx} = 2B', \quad \frac{d\delta y}{dx} + \frac{d\delta x}{dy} = 2C'.$$

A, B, C, a, b, c, being functions of x, y, z, the body will have at each point an infinite number of inextensible directions situated on the cone (real or imaginary),

$$A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma + 2A'\cos\beta\cos\gamma + 2B'\cos\gamma\cos\alpha + 2C'\cos\alpha\cos\beta = 0.$$

If the constitution of the body be given, A, B, C, &c., will be given functions. In this case the equations (D′) furnish the means of determining

δx, δy, δz. Thus, for example, if the body be homogeneous, A, B, C, &c., will be constants, and it is not difficult to prove that δx, δy, δz will be of the form

$$\begin{aligned} \delta x &= ax + by + cz + d, \\ \delta y &= a'x + b'y + c'z + d', \\ \delta z &= a''x + b''y + c''z + d''. \end{aligned} \tag{E'}$$

Let x', y', z', be the co-ordinates of the molecule in its new position. Then since

$$\begin{aligned} \delta x &= x' - x, \\ \delta y &= y' - y, \\ \delta z &= z' - z; \end{aligned}$$

we have

$$\begin{aligned} x' &= (a + 1)\, x + by + cz + d, \\ y' &= a'x + (b' + 1)\, y + c'z + d', \\ z' &= a''x + b''y + (c'' + 1)\, z + d''. \end{aligned} \tag{F'}$$

Hence it is easy to infer the following theorem:

If a homogeneous body have at each point a cone of inextensible directions, and if in the interior of the body there be described an algebraic surface of any order, all the molecules situated upon that surface will after displacement be situated upon a surface of the same order.

In general, whatever be the nature of the body, if ds be an element making with the axes the angles α, β, γ, which satisfy the equation

$$\frac{d\delta x}{dx}\cos^2\alpha + \frac{d\delta y}{dy}\cos^2\beta + \frac{d\delta z}{dz}\cos^2\gamma$$
$$+\left(\frac{d\delta y}{dz} + \frac{d\delta z}{dy}\right)\cos\beta\cos\gamma + \left(\frac{d\delta z}{dx} + \frac{d\delta x}{dz}\right)\cos\alpha\cos\gamma + \left(\frac{d\delta x}{dy} + \frac{d\delta y}{dx}\right)\cos\alpha\cos\beta = 0;$$

it is plain that we shall have

$$\delta ds = 0.$$

Hence,

Whatever be the law of the displacement, there will be at each point of the body an infinite number of directions (forming a cone of the second order), for which the length of the element will be unchanged.

We shall now return to the case of surfaces.

8. The preceding discussion of the properties of inextensible surfaces is of course a mathematical abstraction, not strictly applicable to any substance which we find in nature. Every membrane with which we are acquainted is possessed of some extensibility; and all substances have of course a certain thickness. Our definition, therefore, of an inextensible surface is not strictly true for any really existing substance. But as there are in nature many substances for which this definition is very approximately true, it becomes a question of some interest to determine how far the results of the preceding investigation are applicable to such substances. We shall, therefore, proceed to consider the case of a membrane whose thickness is indefinitely small as compared with its other dimensions, and whose extensibility is such that in any displacement of the membrane, the variation in the *length* of any arc of a curve traced upon its surface is indefinitely small compared with the displacement of any of its parts. Thus, if x, y, z be the co-ordinates of any point on the surface, and s an arc of a curve traced upon it, the assumption which we shall make as to the inextensibility of the membrane may be mathematically expressed by saying that δs is indefinitely small compared with δx. If it be necessary to take *thickness* into account, we must suppose s to be a curve traced arbitrarily in the substance of the membrane. Supposing, for the sake of greater generality, that this is the case, we may state the problem under discussion as follows:

To determine the possible displacement of a membrane very slightly extensible, and whose thickness is very small compared with its other dimensions.

Let x', y', z' be the co-ordinates of a point in the substance of the membrane; x, y, z, the co-ordinates of a point on the surface, indefinitely near to the first; and i, a quantity of the same order of magnitude as the thickness of the membrane.

Through the point $x'y'z'$ let a normal be drawn to the surface of the membrane, and let n represent the part of the normal between $x'y'z'$ and its intersection with the surface, which we shall denote by x, y, z. Then if α, β, γ be the cosines of the angles which the normal makes with the axes, we shall have

$$\begin{aligned} x' &= x + \alpha n, \\ y' &= y + \beta n, \\ z' &= z + \gamma n. \end{aligned} \qquad (G')$$

Differentiating these equations, and rejecting $nd\alpha$, $nd\beta$, $nd\gamma$, on account of the small quantity n, which is of the same order as the thickness of the membrane, we have

$$\begin{aligned} dx' &= dx + \alpha dn, \\ dy' &= dy + \beta dn, \\ dz' &= dz + \gamma dn. \end{aligned} \tag{H'}$$

If now we represent by ds' an arc of a curve traced in the substance of the membrane, we shall have, as before,

$$ds'\delta ds' = dx' d\delta x' + dy' d\delta y' + dz' d\delta z'. \tag{I'}$$

But if we regard $\delta x'$, $\delta y'$, $\delta z'$ as functions of the three variables x, y, n, we shall have

$$\begin{aligned} d\delta x' &= \frac{d\delta x'}{dx}\, dx + \frac{d\delta x'}{dy}\, dy + \frac{d\delta x'}{dn} dn, \\ d\delta y' &= \frac{d\delta y'}{dx}\, dx + \frac{d\delta y'}{dy} dy + \frac{d\delta y'}{dn}\, dn, \\ d\delta z' &= \frac{d\delta z'}{dx}\, dx + \frac{d\delta z'}{dy}\, dy + \frac{d\delta z'}{dn} dn. \end{aligned} \tag{K'}$$

Substituting in (I′) the values (H′) and (K′), and putting for dz its value $pdx + qdy$, we have

$$\begin{aligned} \frac{d\delta s'}{ds} = & \left(\frac{d\delta x'}{dx} + p\,\frac{d\delta z'}{dx}\right)\frac{dx^2}{ds'^2} + \left(\frac{d\delta y'}{dy} + q\,\frac{d\delta z'}{dy}\right)\frac{dy^2}{ds'^2} \\ & + \left(\frac{d\delta x'}{dy} + \frac{d\delta y'}{dx} + p\,\frac{d\delta z'}{dy} + q\,\frac{d\delta z'}{dx}\right)\frac{dx}{ds'}\frac{dy}{ds'} \\ & + \left(\frac{d\delta x'}{dn} + p\frac{d\delta z'}{dn} + \alpha\frac{d\delta x'}{dx} + \beta\frac{d\delta y'}{dx} + \gamma\frac{d\delta z'}{dx}\right)\frac{dx}{ds'}\frac{dn}{ds'} \\ & + \left(\frac{d\delta y'}{dn} + q\frac{d\delta z'}{dn} + \alpha\frac{d\delta x'}{dy} + \beta\frac{d\delta y'}{dy} + \gamma\frac{d\delta z'}{dy}\right)\frac{dy}{ds'}\frac{dn}{ds'} \\ & + \left(\alpha\frac{d\delta x'}{dn} + \beta\frac{d\delta y'}{dn} + \gamma\frac{d\delta z'}{dn}\right)\frac{dn^2}{ds'^2}. \end{aligned} \tag{L'}$$

Now since $d\delta s'$ is by hypothesis indefinitely small, as compared with any one

of the quantities $d\delta x'$, $d\delta y'$, $d\delta z'$, and since this is true for all directions of the arc ds', it is plain that the coefficients of each of the quantities

$$\frac{dx^2}{ds'^2}, \quad \frac{dy^2}{ds'^2}, \quad \frac{dx}{ds'}\frac{dy}{ds'}, \quad \frac{dx}{ds'}\frac{dn}{ds'}, \quad \frac{dy}{ds'}\frac{dn}{ds'}, \quad \frac{dn^2}{ds'^2},$$

must be indefinitely small as compared with δx, δy, δz. We shall, in the first place, consider the coefficients of the first three of these quantities.

If we neglect, as before, quantities of the second order, we may evidently substitute in these coefficients δx, δy, δz, for $\delta x'$, $\delta y'$, $\delta z'$. We shall have then

$$\frac{d\delta x}{dx} + p\,\frac{d\delta z}{dx} = ia,$$

$$\frac{d\delta x}{dy} + \frac{d\delta y}{dx} + p\,\frac{d\delta z}{dy} + q\,\frac{d\delta z}{dx} = 2ib,$$

$$\frac{d\delta y}{dy} + q\,\frac{d\delta z}{dy} = ic,$$

be satisfied, where a, b, c are functions of x and y of the same order of magnitude as δx, δy, δz.

Transforming these equations as in p. 346, we find

$$\frac{du}{dx} - wr = ia,$$

$$\frac{du}{dy} + \frac{dv}{dx} - 2ws = 2ib, \qquad (\mathrm{M}')$$

$$\frac{dv}{dy} - wt = ic.$$

Now it is well known, that such a system of equations may always be satisfied by the values

$$u = u' + iu_1,$$
$$v = v' + iv_1,$$
$$w = w' + iw_1;$$

where u', v', w', satisfy the equations

$$\frac{du'}{dx} - w'r = 0,$$

$$\frac{du'}{dy} + \frac{dv'}{dx} - 2w's = 0,$$

$$\frac{dv'}{dy} - w't = 0.$$

Hence it is plain, that the displacements of a surface which is but slightly extensible will differ from those of an inextensible surface, by quantities which are of the same order of magnitude as the extensibility of the surface. From this it is easy to infer, that all the theorems which are *rigorously* true for an inextensible surface are *approximately* true for a surface possessed of an indefinitely small amount of extensibility.

Let us now consider the coefficients of the quantities

$$\frac{dx}{ds'}\frac{dn}{ds'}, \quad \frac{dy}{ds'}\frac{dn}{ds'}, \quad \frac{dn^2}{ds'^2}.$$

These coefficients give the equations

$$\left.\begin{aligned} \frac{d\delta x'}{dn} + p\frac{d\delta z'}{dn} + \alpha\frac{d\delta x'}{dx} + \beta\frac{d\delta y'}{dx} + \gamma\frac{d\delta z'}{dx} &= iA, \\ \frac{d\delta y'}{dn} + q\frac{d\delta z'}{dn} + \alpha\frac{d\delta x'}{dy} + \beta\frac{d\delta y'}{dy} + \gamma\frac{d\delta z'}{dy} &= iB, \\ \alpha\frac{d\delta x'}{dn} + \beta\frac{d\delta y'}{dn} + \gamma\frac{d\delta z'}{dn} &= iC, \end{aligned}\right\} \qquad (\mathrm{N}')$$

A, B, C being of the same order as $\delta x'$, $\delta y'$, $\delta z'$. Since α, β, γ are independent of n, the third of these equations may be integrated at once. Performing the integration, and supposing the integrals to begin when

$$x' = x, \quad y' = y, \quad z' = z,$$

we have

$$\alpha\delta x' + \beta\delta y' + \gamma\delta z' = \alpha\delta x + \beta\delta y + \gamma\delta z + i\int_0^n C dn. \qquad (\mathrm{O}')$$

Now it is evident that

$$\begin{aligned} \alpha\delta x' + \beta\delta y' + \gamma\delta z' &= \delta n, \\ \alpha\delta x + \beta\delta y + \gamma\delta z &= (\delta n)_0, \end{aligned}$$

denoting by $(\delta n)_0$ the normal displacement of the point on the surface. Equation (O′) becomes, therefore,

$$\delta n = (\delta n)_0 + i\int_0^n C dn.$$

Now the definite integral

$$\int_0^n C dn$$

is evidently a small quantity of the second order; if therefore we neglect quantities of the third order, we shall have

$$\delta n = (\delta n)_0.$$

Hence we infer that—

In all possible displacements of a thin membrane or lamina which is very slightly extensible, the normal displacements of points situated on the same normal to the surface are equal.

This would also follow from the next theorem.

Substituting in the first two equations (N′) for α, β, γ their values in terms of p and q, we have

$$\left.\begin{aligned} &\frac{d\delta x'}{dn} + p\frac{d\delta z'}{dn} + \frac{1}{\sqrt{(1+p^2+q^2)}}\left(\frac{d\delta z'}{dx} - p\frac{d\delta x'}{dx} - q\frac{d\delta y'}{dx}\right) = iA, \\ &\frac{d\delta y'}{dn} + p\frac{d\delta z'}{dn} + \frac{1}{\sqrt{(1+p^2+q^2)}}\left(\frac{d\delta z'}{dy} - p\frac{d\delta x'}{dy} - q\frac{d\delta y'}{dy}\right) = iB. \end{aligned}\right\} \quad (\mathrm{P}')$$

Now it is plain that without altering the form of these equations we may substitute, in the last three terms of each, δx, δy, δz for $\delta x'$, $\delta y'$, $\delta z'$. For this substitution merely amounts to the addition of quantities of the same order as iA, iB, to the right-hand members of these equations. Again, referring to p. 348, we have

$$\frac{d\delta z}{dx} - p\frac{d\delta x}{dx} - q\frac{d\delta y}{dx} = \delta p,$$

$$\frac{d\delta z}{dy} - p\frac{d\delta x}{dy} - q\frac{d\delta y}{dy} = \delta q.$$

Making these substitutions in equations (P′), we have

$$\frac{d\delta x'}{dn} + p\frac{d\delta z'}{dn} + \frac{\delta p}{\sqrt{(1+p^2+q^2)}} = iA,$$
$$\frac{d\delta y'}{dn} + q\frac{d\delta z'}{dn} + \frac{\delta q}{\sqrt{(1+p^2+q^2)}} = iB.$$

Integrating these equations between the limits 0 and n, and neglecting as before quantities of the third order, we have

$$\left.\begin{aligned} &\delta(x'-x) + p\delta(z'-z) + \frac{n\delta p}{\sqrt{(1+p^2+q^2)}} = 0, \\ &\delta(y'-y) + q\delta(z'-z) + \frac{n\delta q}{\sqrt{(1+p^2+q^2)}} = 0. \end{aligned}\right\} \quad (\mathrm{Q}')$$

But since

$$\frac{n}{\sqrt{(1+p^2+q^2)}} = z' - z,$$

these equations may evidently be written

$$\delta\{x' - x + p(z'-z)\} = 0,$$
$$\delta\{y' - y + q(z'-z)\} = 0.$$

Hence recollecting that the points xyz, $x'y'z'$, were originally on the same normal, we have still, *after* the displacement,

$$x' - x + p(z'-z) = 0,$$
$$y' - y + q(z'-z) = 0.$$

We infer, therefore, that—

In every possible displacement of a thin membrane or lamina whose extensibility is very small, all points which were originally situated on the same normal to the surface will remain so after the displacement.

This important theorem, which is assumed as an hypothesis by most writers on the equilibrium of elastic laminæ, is thus established, independently of any theory of molecular force, as a mathematical consequence of the small amount of extensibility which is possessed by the lamina.

It may be well, before concluding, to say a few words in explanation of the rule which we have followed in the rejection of small quantities.

Small quantities of the first order, as δx, &c., have been retained throughout.

Small quantities of the second order are rejected in the expressions for dx', dy', dz', (H′), because the retention of these quantities would leave the form of the equations (M′) and (N′) altogether unchanged.

Small quantities of the third order are rejected in equations (O′) and (Q′), because the differences

$$\delta n - (\delta n)_0, \quad \delta x' - \delta x, \text{ \&c.}$$

ought properly to be of the second order. If we retain these terms we may enunciate the foregoing theorem rigorously as follows :

If a membrane which is but slightly extensible receive a finite displacement, the separation of any point from the normal drawn through the corresponding point on the surface, is indefinitely small, as compared with the distance of these points from each other.

With respect to the comparative magnitude of the two small quantities i and n, depending respectively upon the extensibility and the thickness of the lamina, it may have been observed that throughout the preceding discussion they have been treated as quantities of the same order. Let us consider what would be the effect of a violation of this rule.

As the thickness of the membrane is not supposed to be insensible, we cannot suppose n to be indefinitely small as compared with i, without assigning to the membrane an amount of extensibility *not* indefinitely small. This would remove it from the class of substances which we have been considering.

If we had supposed i to be indefinitely small as compared with n, we should not have been justified in rejecting $nd\alpha$, $nd\beta$, $nd\gamma$ in forming the expressions (H′), p. 368. Our investigation would not, therefore, have differed in any material respect from that of the displacements of a *body* of finite dimensions and of an indefinitely small amount of extensibility; and in such a case it would readily appear from the discussion of p. 365 that the body would be, q. p., rigid. We see then that—

No membrane can be flexible which does not possess an amount of extensibility finite, as compared with its thickness.

It is, perhaps, superfluous to add, that it is not necessary to the truth of the preceding theorems that the membrane should be absolutely or approximately inextensible by any imaginable force. It is sufficient for our purpose if the

forces which are supposed to be applied to the membrane are incapable of extending it. And in such a case all the foregoing theorems will hold, if we substitute for "all possible displacements," "all displacements which can be effected by any amount of force which is supposed to be present."

Some interesting practical conclusions follow from this discussion. Thus, if we desire to take advantage of the very slight extensibility of many species of laminæ, to enable them to resist flexure, it appears, from p. 359, that we must be careful to form the lamina originally, while in a soft, semi-fluid, or otherwise extensible state, into a surface whose curvatures are similar, otherwise it will always be liable to bend along a curve of flexure. If sufficient force be used to make the lamina bend along any other curve or in any way violate the conditions which have been established, it will be found that there is always produced a *crease*, in other words a curve, along which the separation between one molecule and the next is not indefinitely small. In such a case there will in general be a permanent alteration in the substance of the lamina. Thus, for example, it is easy to fold a sheet of paper into the form of a cone, without breaking or in any way injuring it. Let the base of this cone be rendered rigid by being attached to a ring, and it will be found that any further attempt to bend the paper will produce a *crease*, or curve of permanent alteration in its substance.

Again, from the discussion of p. 361, we may deduce the practical conclusion, that the strength by which a surface of dissimilar curvatures resists flexure may be greatly increased, if it be traversed by a rigid rod attached to its substance, along any curve not a curve of flexure.

NOTE.

SINCE the foregoing sheets were printed, I have arrived at the following theorem, which is of some interest, as connected with the class of surfaces which we have been examining:

If a closed oval surface be perfectly inextensible, it is also perfectly rigid.

To prove this, let us denote, as before, by δx, δy, δz, the resolved displacements of any point on the surface. Let $\delta' x, \delta' y, \delta' z$ be its most general displacements considered as a rigid body; then it is known that

$$\begin{aligned} \delta' x &= a + Cy - Bz, \\ \delta' y &= b + Az - Cx, \\ \delta' z &= c + Bx - Ay, \end{aligned}$$

a, b, c, A, B, C, being constants. Now if we form a third system—

$$\begin{aligned} \Delta x &= \delta x + \delta' x, \\ \Delta y &= \delta y + \delta' y, \\ \Delta z &= \delta z + \delta' z, \end{aligned}$$

it is plain that Δx, Δy, Δz will satisfy the conditions of the problem contained in equations (B) or (C). Moreover, if $x_1 y_1 z_1$, $x_2 y_2 z_2$ be two given points on the surface, the constants a, b, c, A, B, C, can always be so determined as to satisfy the equations

$$\begin{aligned} \Delta x_1 = 0, \quad \Delta z_1 = 0, \\ \Delta x_2 = 0, \quad \Delta z_2 = 0, \end{aligned}$$

without in any way limiting the generality of the displacements δx, δy, δz. Suppose now that we assume, as in p. 346,

$$u = \Delta x + p \Delta z, \quad w = \Delta z,$$

it is plain that u, w will satisfy the first of equations (C), and will vanish at the two points $x_1 y_1 z_1$, $x_2 y_2 z_2$. Let these points be P, Q, and suppose, to fix our ideas, that the axis of z passes through them. The plane of xz will then intersect the surface in a closed curve, $PRQS$, passing through these points. Now since u vanishes at the points P, Q, if we trace its values in passing along the curve $PRQS$, we shall find a *maximum* value (disregarding its sign) somewhere between P and Q as at R, and again somewhere between Q and P as at S. We have, therefore, for each of the points R, S,

$$\frac{du}{dx} = 0,$$

since the equation of the curve $PRQS$ is

$$dy = 0.$$

The first of equations (C) gives us then at each of these points

$$w = 0.$$

But since the position of the axis of x is indeterminate, it follows from what has been said, that, on *every* section of the surface made by a plane passing through the axis of z, there will be at least two points, for which

$$w = 0.$$

Hence it is plain that there will be on the surface one or more closed curves for which this condition will hold. It will be sufficient to consider one of these curves, which, for the sake of distinctness, we may call an equator.

We have seen, p. 347, that w must satisfy the equation

$$r\frac{d^2w}{dy^2} - 2s\frac{d^2w}{dxdy} + t\frac{d^2w}{dx^2} = 0,$$

or, as it may be otherwise written,

$$\frac{d}{dy}\left(r\frac{dw}{dy} - s\frac{dw}{dx}\right) + \frac{d}{dx}\left(t\frac{dw}{dx} - s\frac{dw}{dy}\right) = 0.$$

Multiply this equation by $dxdy$, and integrate it through the whole of either of the segments into which the surface is divided by the equator. We have then

$$\int\left(r\frac{dw}{dy} - s\frac{dw}{dx}\right)dx + \int\left(t\frac{dw}{dx} - s\frac{dw}{dy}\right)dy = 0,$$

the single integrations being extended through the whole of the bounding curve. But since, for every point in this curve we have

$$w = 0,$$

if this equation be transformed according to the usual rule (*Calculus of Variations*, p. 218) it will become

$$\int\left(r\frac{dw^2}{dy^2} - 2s\frac{dw}{dy}\frac{dw}{dx} + t\frac{dw^2}{dx^2}\right)\Omega ds = 0,$$

where ds is the element of the bounding curve, and

$$\Omega = \left(\frac{dw^2}{dx^2} + \frac{dw^2}{dy^2}\right)^{-\frac{1}{2}}$$

Now since in the class of surfaces which we are considering,

$$rt - s^2 > 0,$$

it is easily seen that all the elements of the foregoing definite integral must have the same sign. The total integral cannot therefore vanish unless each of its elements vanishes. Hence it is plain that we must have at each point of the equator

$$\frac{dw}{dx} = 0, \quad \frac{dw}{dy} = 0.$$

If we now follow the same reasoning as in p. 352 we shall readily see that *all* the differential coefficients of w will vanish at the equator, and therefore that we must have *generally*

$$w = 0.$$

Hence, and from p. 347, it is evident that the displacements represented by Δx, Δy, Δz, are those of a rigid body. Since then $\delta' x$, $\delta' y$, $\delta' z$ are by hypothesis the displacements of a rigid body, it is evident that the differences between these quantities, $\Delta x - \delta' x$, $\Delta y - \delta' y$, $\Delta z - \delta' z$, or δx, δy, δz, are so likewise. We infer, therefore, that—

The most general displacement which a closed, oval, inextensible surface admits of, is that of a rigid body.

Such a surface is therefore inflexible.